PRIMES Circle 2022 - Introduction to Knot Theory

Pete Olhava, Alex Gil Osorio

May 21 2022

1 Introduction

This paper follows [Ada94].

Knot theory is the study of closed curves suspended in three dimensional space and how they can be deformed and categorized without passing through itself. One of the most basic knots is an unknot, which in its most basic form is simply a circle. The common definition of knots, like tying your shoelace isn't the same as knots in knot theory.

Definition 1.1. *A* ***knot*** *is a loop of string with no thickness, and is a closed curve in 3D space.*

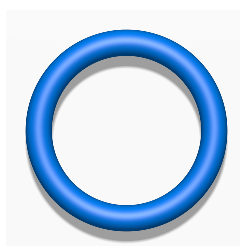

Figure 1: This is the unknot.

[**?**]

Another common knot, (see figure 2) is the trefoil which is the second most simplest knot in it's basic form. To study knots, it is convenient to work with drawings of knots in a 2D plane called projections, so we will define projections of knots.

Definition 1.2. *A* ***projection*** *is an image of a knot.*

There are infinite projections of each knot, because knots are actually tangled loops in 3D. When we show them on paper they are just a picture of a knot from

Figure 2: These are projections of the unknot (left) and trefoil (right).

a certain angle, while the knot is tangled in a certain way. The ways that we can switch between projections in 2D are called planar isotopy and reidemeister moves.

Definition 1.3. ***Planar Isotopy*** *is the deformation of a knot within the projection plane.*

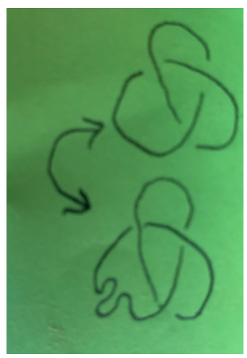

Figure 3: Example of planar isotopy in a trefoil.

There are three other ways to deform a knot and these moves are called Reidemeister moves.

Definition 1.4. ***Reidemeister moves*** *are the the three ways to change the crossings of a knot, while keeping the knot the same.*

These moves look like this.

Any projection of a knot can be transformed into any other projection of that same knot using only planar isotopy and reidemeister moves.

Definition 1.5. *A* ***link*** *is any number of knots that are interconnected, a single knot is also a link.*

Distinguishing links can be hard but you can use link invariants like **tricolorability** to make it easier. Tricolorbility is an invariant because a link is either tricolorable or not.

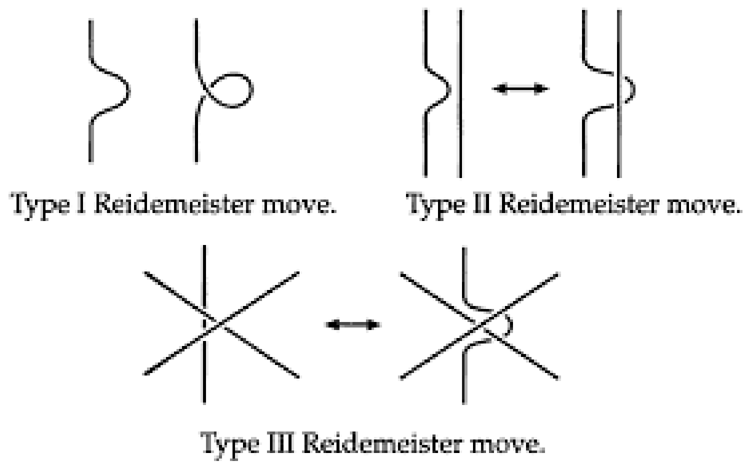

Figure 4: The three Reidemeister moves.

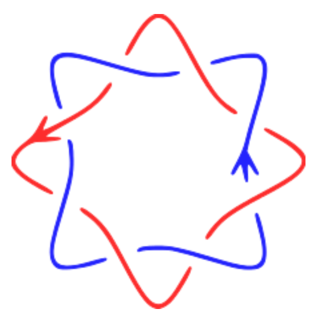

Figure 5: This is an example of a link with 2 knot components, one knot is red and the other is blue.

Definition 1.6. *In the mathematical field of knot theory, the tricolorability of a knot is the ability of a knot to be colored with three colors subject to certain rules. Tricolorability is an isotopy invariant, and hence can be used to distinguish between two different knots.*

Because reidemeister moves and planar isotopy preserve tricolorability on a projection we can prove that the trefoil and unknot are two different knots because the projections of the unknot are not tricolorable while the projections of the trefoil are.

Definition 1.7. ***Orientation*** *is the direction that we choose to travel around a knot. To show this we place arrows on the knot.*

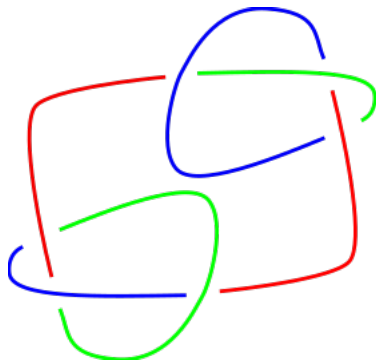

Figure 6: A tricolorable projection.

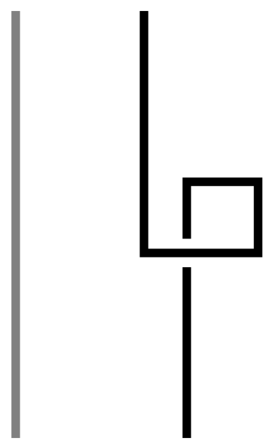

Figure 7: Reidemeister I move preserves tricolorability

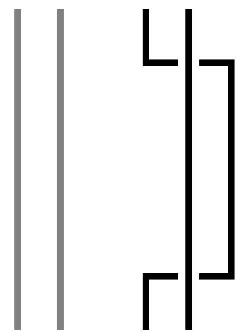

Figure 8: Reidemeister II move preserves tricolorability

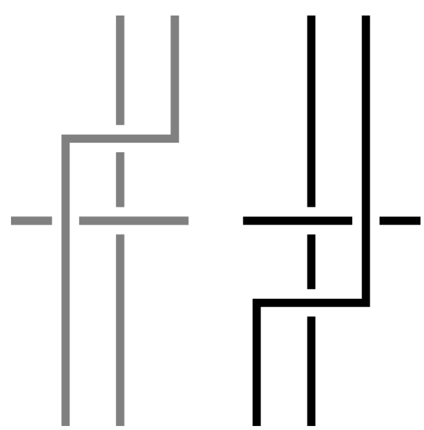

Figure 9: Reidemeister III move preserves tricolorability

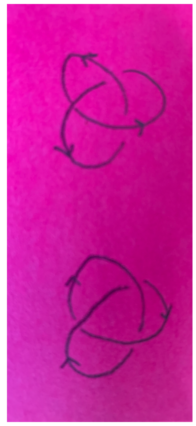

Figure 10: These are the two ways to orient a trefoil knot.

2 Biology

You may being wondering if we can apply knot theory to anything in our lives, well we can. At first people thought that knot theory explained the structure of atoms, but that hypothesis was proved to be incorrect. One application for knot theory however is right inside you, your DNA! DNA or deoxyribonucleic acid is formed in a structure called a **double helix.**

Definition 2.1. ***Double helix's*** *are a pair of helix's intertwined around a line (see figure 11).*

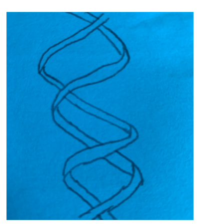

Figure 11: A double helix.

Definition 2.2. *Enzymes called* ***topoisomerases*** *can changes the topology of DNA.*

If the DNA that the topoisomerases alter is in a loop then these enzymes would also change the type of knot or link the DNA is configured in. There are three main ways these enzymes are able to change the topology of DNA (see figure 12).

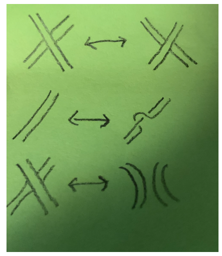

Figure 12: Most common actions for enzymes to take on DNA.

These aren't the only actions enzymes can take on DNA, certain enzymes are able to create more complex changes to the topology of DNA.

Definition 2.3. ***Cyclic DNA*** *is a category of DNA in which it is shaped in a closed loop.*

However some DNA is not circular, this makes it troublesome for scientists to discern what effect an enzyme would have on the topology of the DNA. But circular DNA models shows how an enzyme knots the DNA and doesn't allow the knot to simply slip off the end of the strand.

What is the purpose of topoisomerases? DNA is all tangled up in the cell, which makes it difficult for the cell to carry out its functions. Topoisomerases are able to manipulate topological structure of the DNA to allow the cell to "see" its DNA and know what to do.

Because cyclic DNA forms a closed loop it can be classified as a knot. Knot theory can be used to study cyclic DNA by modeling it as a ribbon.

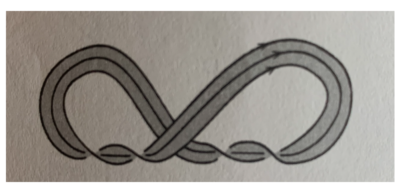

Figure 13: Ribbon model of cyclic DNA with an orientation.

If we treat this ribbon as fixed in its current position in 3D space we are able to find invariants for the ribbon. To calculate the twist of a ribbon which is called, Tw(R) we measure the amount of times the ribbon crosses over its axis. Tw(R) is equal to one half of the sum of the crossings numbers. You can find these numbers by using this diagram (see figure 14).

We also must calculate the writhe of the ribbon. Wr(r). To do this we need to find the signed crossover number or the sum of all ± 1s on the crossings. And we have to do this for any projection of the ribbon. We can use this equation

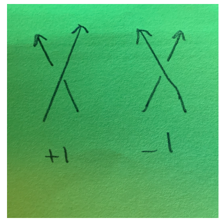

Figure 14: How to calculate the linking number of a crossing.

to find the writhe of a ribbon. To calculate the linking number of a ribbon or Lk(R) we add Tw(R) and Wr(R).

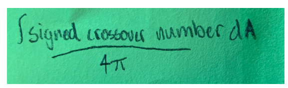

Figure 15: Equaation to calculate writhe.

DNA phosphates (parts that make up the larger DNA structure) have a way of orientating themselves too, just like other knots. Each molecule has a 5' and a 3' prime end and when connecting with each other a 5' must touch a 3', it may be helpful to think of this as Lego bricks connecting, where you need the protruding circles on the top to connect to the indentations on the bottom of the Lego. However this only works on cyclic DNA because linear DNA will have a 5' and 3' end on either side, thus given us no way to orient it.

Figure 16: Primes.

References

[Ada94] Colin Adams, *The knot book: An elementary introduction to the mathematical theory of knots*, W. H. Freeman, New York, 1994.

wikepedia.com